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STRESS FIELD OF PLANE DISLOCATION PILE-UPS IN ANISOTROPIC THEORY OF ELASTICITY

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The stresses at each point of an infinite anisotropic elastic medium with defects described effectively by dislocation pile-ups (the pile-ups themselves, cracks, deformation twins, fine crystals of martensitic type) are determined within the framework of the plane problem of the theory of elasticity. The stresses are expressed in terms of the dislocation density in the pile-ups for an arbitrary realization of the defects, taking into account their total Burgers vector. The interaction between a source of the linear force-dislocation type and the mentioned defects (taking account of the image force) is investigated. An expression is presented for the stress around pile-ups described discretely.

A number of papers ([1 — 7], for example) is devoted to a theoretical study of the stress fields of defects of pile-up type. Attention is turned in some [4 — 7] to the similarity between the type of stress and the dislocation distribution in the pile-up. Nevertheless, no general expression has been obtained for the stress field in an anisotropic elastic and even isotropic medium with pile-ups although there is a solution for a slit-like crack [1] (valid, however, only in particular cases).

By using a simpler method than in [1], more exact results are obtained in this paper which are applicable to all defects of the plane dislocation pile-up type for an arbitrary realization of the defects. In contrast to the expressions in [1] which are valid only near the blocked ends of cracks with zero Burgers vector and do not accurately take account of the interaction with other defects, the expressions obtained yield the stress in a whole medium with defects of the pile-up type in external fields which can be described by linear combinations of rational-fraction functions.

Let us note that the influence of anisotropy on the dislocation distribution and on the stress, not essential in a number of cases [1, 2], can play a major part in describing such processes as the excitation of defects by definite development systems in a field of others, the rotation and branching of cracks, the distribution of impurities and the separation of phases around defects, the formation of relaxation zones around them, etc.

1. The stress field at a point (x, y) near a dislocation parallel to Oz and intersecting the xy-plane at the point $(\xi, 0)$ is

$$\sigma_{kj}^{(d)}(\xi;\, x,\, y) = \sum_{\nu=1}^{6} \frac{Q_{kj}^{(\nu)}}{\zeta_\nu - \xi}, \quad \zeta_\nu = x + p_\nu y \tag{1.1}$$

Here the complex constants $Q_{kj}^{(\nu)}$ and p_ν ($\operatorname{Im} p_\nu \neq 0$ [8]) depend on the dislocation direction and its Burgers vector and the elastic properties of the medium (the method of their evaluation is elucidated in [1], in Chap. 13 of the book [8], and in the appendix).

The dislocation distribution $\rho(x)$ in the pile-up on the section $\Delta_1 \leqslant x \leqslant \Delta_2$ of the xy-plane is determined in terms of the external stresses by inverting the singular integral equation

$$\int_{\Delta_1}^{\Delta_2} \frac{\rho(\xi)\,d\xi}{\xi - x} = A^{-1}\sigma_{kj}n_k b_j, \quad A = \sum_{\nu=1}^{6}{}' Q_{kj}^{(\nu)} n_k b_j \tag{1.2}$$

Here A is the constant of dislocation interaction with the Burgers vector $\mathbf{b}$ in a pile-up on a plane with normal $\mathbf{n}$ (in this case the vector $\mathbf{n}$ is directed along Oy).

According to [9], the solution of (1.2) is

$$\rho(x) = -\frac{1}{\pi^2}\sqrt{R(x)}\left[C + \int_{\Delta_1}^{\Delta_2} \frac{\sigma(\xi)\,d\xi}{\sqrt{R(\xi)}\,(\xi - x)}\right] \tag{1.3}$$

In this expression $R(x)$ is a rational fraction whose numerator is the product of those quantities $x - \Delta_m$ $(m = 1, 2)$, where Δ_m are the free ends of the pile-up, and the product of those $x - \Delta_n$, where Δ_n are the blocked ends, is in the denominator, and C is a constant which is not zero only for pile-ups blocked on two sides. The condition

$$\int_{\Delta_1}^{\Delta_2} \rho(\xi)\,d\xi = N \tag{1.4}$$

defining the constant C or the position of the end of the defect (N is the total number of dislocations in the defect) is satisfied in all cases. If both ends are free, compliance with the additional condition

$$\int_{\Delta_1}^{\Delta_2} \frac{\sigma(\xi)\,d\xi}{\sqrt{R(\xi)}} = 0 \tag{1.5}$$

predetermining the position of the ends of the defect, is required. The anisotropy enters only through the constant A in the expressions written down.

If cohesive forces act at the ends of the defects or there is a plastic relaxation zone, they are often considered clamped, but occupying a position such that the coefficient in the root singularity would not exceed a given number (see the end of Sect. 3).

It is easy to extend (1.2) — (1.4) etc. to the case when there are sections with zero dislocation density in the pile-ups [9] (i.e. to the case of several pile-ups in one plane). Then the integral in (1.2) and (1.3) must be replaced by a sum of s integrals over the sections $[\Delta_{k1}, \Delta_{k2}]$ $(k = 1, 2, \ldots, s)$ with nonzero functions $\rho(x)$; furthermore,

$$R = R_1(x)R_2(x)\ldots R_s(x),$$

where $R_k(x)$ is determined for the k-th section as before; $C_0 + C_1 x + \ldots + C_{q-1}x^{q-1}$ must be substituted in place of the constant C; condition (1.4) is replaced by s conditions (1.6), and conditions (1.5) by condition (1.7)

$$\int_{\Delta_{k1}}^{\Delta_{k2}} \rho(\xi)\,d\xi = N_k \tag{1.6}$$

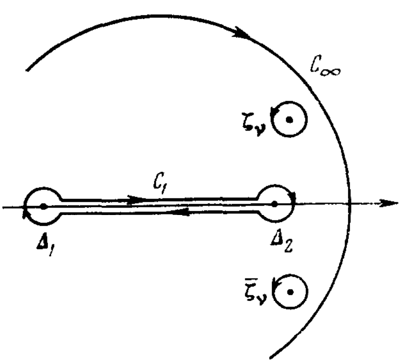

Fig. 1

$$\sum_{k=1}^{s}\int_{\Delta_{k1}}^{\Delta_{k2}} \frac{\xi^m\sigma(\xi)\,d\xi}{\sqrt{R(\xi)}} = 0, \quad m = 0, 1, \ldots, -q-1 \tag{1.7}$$

(q is the difference between s and the number of free ends).

The stress field around a pile-up is given by the integral

$$\sigma_{kj}(x, y) = \int_{\Delta_1}^{\Delta_2} \rho(\xi)\, \sigma_{kj}^{(d)}(\xi;\, x, y)\, d\xi$$

Let us transform this formula by using analytic continuation of the function $\rho(x)$ from the upper edge of $[\Delta_1, \Delta_2]$ onto the whole complex plane

$$\rho(\zeta) = \frac{1}{\pi i}\left[\Phi(\zeta) - \sigma(\zeta)\right], \quad \Phi(\zeta) = \int_{\Delta_1}^{\Delta_2} \frac{\rho(\xi)\, d\xi}{\xi - \zeta} \tag{1.8}$$

Here $\Phi(\zeta)$ is a function of the complex variable ζ which is analytic outside the slit $[\Delta_1, \Delta_2]$ in the case of integrability of $\rho(x)$ and $\sigma(\zeta)$ is the analytic continuation of $\sigma(x)$ which is meromorphic if only $\sigma(x)$ defined by (1.2) is a rational function. Since the values of $\Phi(\zeta)$ at the edges of the slit satisfy the Sokhotskii-Plemelj relations, we have at points of continuity of $\rho(x)$

$$\rho(x + i0) = \rho(x) = -\rho(x - i0)$$

Hence, the stresses can be written as the sum of contour integrals (the contour of integration is shown in Fig. 1)

$$\sigma_{kj}(x, y) = \sum_{\nu=1}^{6} \frac{1}{2} Q_{kj}^{(\nu)} \oint \frac{\rho(\zeta)\, d\zeta}{\zeta_\nu - \zeta}$$

The integrand has just the following singularities outside the contour: a pole at the point $\zeta_\nu = x + p_\nu y$ (which does not lie on $[\Delta_1, \Delta_2]$ because $\operatorname{Im} p_\nu \neq 0$), singularities of the functions $\sigma(\zeta)$ and possibly a pole at infinity. Hence, the desired stress field around the pile-up is given by the expression

$$\sigma_{kj}(x, y) = \sum_{\nu=1}^{6} Q_{kj}^{(\nu)} \left\{-\pi i \rho(x + p_\nu y) + \sum_{(\sigma)} \operatorname{Res}\left[\frac{\sigma(\zeta)}{\zeta - x - p_\nu y}\right]\right\} \tag{1.9}$$

in which the second summation is taken over the singularities of $\sigma(\zeta)$ and the infinitely distant point.

The condition for the possibility of evaluating the residues of $\sigma(\zeta)$ imposes a constraint of the kind of admissible load. The presence of component of the stress at infinity results in a pole $\sigma(\zeta)/\zeta$ there and the appearance of the difference $\rho(\zeta_\nu) - \rho(\infty)$ in the right side of (1.9). In the case of several pile-ups in one plane, (1.9) remains valid, taking into account the modification of $\rho(x)$ discussed earlier.

Let us stress the need to select that branch of the function $\rho(\zeta)$ which is the continuation of $\rho(x)$ from the upper edge of the slit $[\Delta_1, \Delta_2]$. A number of expressions for ρ is determined by the function $\sqrt{(\Delta_2 - \zeta)(\zeta - \Delta_1)}$; the needed branch of this function tends to $-i\zeta$ at infinity.

Let us present expressions for ρ and σ_{kj} in the cases encountered most often. We have for a pile-up of N dislocations clamped at the points Δ_1 and Δ_2 in the field $\sigma_0 - \alpha x$

$$\rho(x) = \frac{8N + 4\sigma_0(2x - \Delta_1 - \Delta_2) - \alpha\left[8x^2 - 4x(\Delta_1 + \Delta_2) - (\Delta_2 - \Delta_1)^2\right]}{8\pi\sqrt{(\Delta_2 - x)(x - \Delta_1)}} \tag{1.10}$$

$$\sigma_{kj}(x, y) = \sum_{\nu=1}^{6} Q_{kj}^{(\nu)}\left\{-\pi i \rho(\zeta_\nu) + \sigma_0 - \alpha\zeta_\nu\right\}$$

By imposing the condition of boundedness of $\rho(x)$ at the free ends and the condition (1.5), we obtain formulas from these expressions, for a pile-up drawn into a detainer at Δ_2 ($\alpha = 0$, $\Delta_2 - \Delta_1 = 2N/\sigma_0$)

$$\rho(x) = \frac{\sigma_0}{\pi}\sqrt{\frac{x-\Delta_1}{\Delta_2 - x}} \tag{1.11}$$

$$\sigma_{kj}(x, y) = \sum_{\nu=1}^{6} Q_{kj}^{(\nu)}\{-\rho(\zeta_\nu) + \sigma_0\}$$

as well as for a free pile-up in a quadratic potential well ($\sigma_0 = 0$, $\Delta_2 = -\Delta_1 = \Delta = \sqrt{2N/\alpha}$)

$$\rho(x) = \frac{\alpha}{\pi}\sqrt{\Delta^2 - x^2}, \qquad \sigma_{kj}(x, y) = \sum_{\nu=1}^{6} Q_{kj}^{(\nu)}\left(-i\alpha\sqrt{\Delta^2 - \zeta_\nu^2} - \alpha\zeta_\nu\right) \tag{1.12}$$

Some other examples are discussed in Sect. 3.

The expression (1.9) is applicable for arbitrary dislocation pile-ups, particularly twinning [10] or transformation dislocations [11, 12], and therefore, they determine the stresses around fine twins or martensitic crystallites (it is essential only that the crystallite be sufficiently fine and the difference between the elastic deformation of the crystallite and the matrix could be neglected in the stress field originating as compared with the intrinsic transformation deformation; only the elastic moduli of the matrix hence enter into the considerations).

2. A slit-like crack (extended along Oz) is described [2, 6] by a system of three families of dislocations: (1) edge (cleavage) dislocation with Burgers vector perpendicular to the plane of the crack (it is convenient to select it as the unit $\mathbf{b}^{(2)} = \mathbf{n}$); (2) screw shear (with Burgers vector parallel to the front of the crack $\mathbf{b}^{(3)} = \boldsymbol{\tau}$), and (3) edge shear (Burgers vector in the plane of the crack and perpendicular to the front $\mathbf{b}^{(1)} = \mathbf{m}$). In this case

$$\sigma_{kj}(x, y) = \sum_{\alpha=1}^{3}\sum_{\nu=1}^{6} Q_{kj}^{(\alpha\nu)}\left\{-\pi i \rho^{(\alpha)}(\zeta_\nu) + \sum_{(\sigma)} \operatorname{Res}\left[\frac{\sigma^{(\alpha)}(\zeta)}{\zeta - \zeta_\nu}\right]\right\} \tag{2.1}$$

(the summation over α is over the three families of crack dislocations).

The dislocation densities $\rho^{(\alpha)}(x)$ in the generally different sections $[\Delta_1^{(\alpha)}, \Delta_2^{(\alpha)}]$ are given by the solution of the following system

$$\sum_{\beta=1}^{3} A^{(\alpha\beta)} \int_{\Delta_1^{(\beta)}}^{\Delta_2^{(\beta)}} \frac{\rho^{(\beta)}(\xi)\, d\xi}{\xi - x} = \tau^{(\alpha)}(x) \qquad (\alpha = 1, 2, 3) \tag{2.2}$$

where

$$A^{(\alpha\beta)} = \sum_{\nu=1}^{6} Q_{kj}^{(\beta\nu)} n_k b_j^{(\alpha)} \tag{2.3}$$

$$\tau^{(\alpha)} = \sigma_{kj} n_k b_j^{(\alpha)} \tag{2.4}$$

The location of the ends of the intervals is found from conditions (1.4) and (1.5) for each of the dislocation families. The pile-ups of shear dislocations can emerge beyond the limits of the cleavage pile-ups, then they become pile-ups of real dislocations. It is assumed in (2.2) that these pile-ups lie in the plane of the crack. The matrix $A^{(\alpha\beta)}$ is non-singular and has the inverse $(A^{-1})^{(\alpha\beta)}$. Hence, in the case of clamped ends

$\Delta_1^{(\alpha)} = \Delta_1,\ \ \Delta_2^{(\alpha)} = \Delta_2$ the solution (2.2) has the form

$$\rho^{(\alpha)}(x) = -\frac{1}{\pi^2}\frac{1}{\sqrt{(\Delta_2 - x)(x - \Delta_1)}}\times \qquad (2.5)$$

$$\left\{C^{(\alpha)} + \int_{\Delta_1}^{\Delta_2}\frac{\sigma^{(\alpha)}(\xi)\sqrt{(\Delta_2-\xi)(\xi-\Delta_1)}\,d\xi}{\xi - x}\right\}$$

$$\sigma^{(\alpha)}(x) = \sum_{\beta=1}^{3}(A^{-1})^{(\alpha\beta)}\tau^{(\beta)}(\xi)$$

($\alpha = 1, 2, 3$; $C^{(\alpha)}$ are determined from the three conditions (1.4)).

In this case $C^{(\alpha)}=0$ (which corresponds to a zero Burger vector for the crack) and the expressions written down are analogous to formulas (102) and (103) in [1] when only the first member on the right side of (2.1) is taken into account.

3. Let us examine the important problem of the interaction between a pile-up and a linear source (which is simultaneously a dislocation and a linear force). The stress field of a source passing through (x_0, y_0) is

$$\sigma_{kj}^{(s)}(x, y) = \sum_{\nu=1}^{6}\frac{U_{kj}^{(\nu)}}{\zeta_\nu - \zeta_{0\nu}} \qquad (3.1)$$

Here $\zeta_{0\nu} = x_0 + p_\nu y_0$. The specific expression for $U_{kj}^{(\nu)}$ is given by (A.8) of the appendix obtained in [13, 14].

The dislocation density in the pile-up with clamped ends $(-\Delta, 0)$, $(\Delta, 0)$ is determined by the expression

$$\rho(x) = \frac{1}{\sqrt{\Delta^2 - x^2}}\left\{C - \frac{n_k b_j}{2\pi^2 A}\oint_{C_2}\sigma_{kj}^{(s)}(\zeta, \zeta_0)\frac{\sqrt{\Delta^2-\zeta^2}}{\zeta - x}\,d\zeta\right\}$$

Integration is along the contour C_2 shown in Fig. 2. The integrals are easily evaluated and we obtain by using (1.4)

$$\rho(x) = \frac{NA + A_{12}}{\pi A\sqrt{\Delta^2 - x^2}} + \frac{1}{\pi i A}\sum_{\nu=1}^{6}\frac{\omega_\nu\sqrt{\Delta^2-\zeta_{0\nu}^2}}{(\zeta_{0\nu} - x)\sqrt{\Delta^2 - x^2}} \qquad (3.2)$$

$$\omega_\nu = U_{kj}^{(\nu)} n_k b_j, \quad A_{12} = \sum_{\nu=1}^{6}\omega_\nu \qquad (3.3)$$

In this case, according to (1.9) the stress field of the pile-up is

$$\sigma_{kj}(x, y) = \sum_{\mu=1}^{6} Q_{kj}^{(\mu)}\left\{\frac{-i(NA + A_{12})}{A\sqrt{\Delta^2 - \zeta_\mu^2}} + \sum_{\nu=1}^{6}\frac{\omega_\nu}{A(\zeta_{0\nu} - \zeta_\mu)}\left[1 - \frac{\sqrt{\Delta^2 - \zeta_{0\nu}^2}}{\sqrt{\Delta^2 - \zeta_\mu^2}}\right]\right\} \qquad (3.4)$$

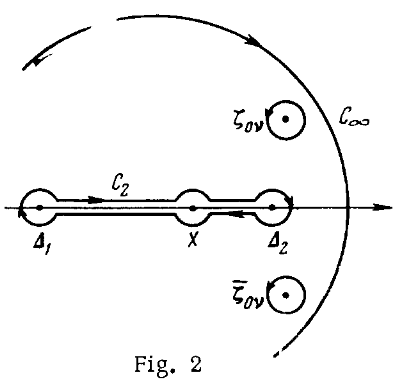

Fig. 2

The expressions (3.2) and (3.4) extend the results in [15], in which the interaction between a crack blocked at two sides for $N_\alpha = 0$ and a dislocation is examined.

The following singularities in the stress (3.4) are seen: 1°. At large distances from the pile-up the field behaves as the field of a N-tuple dislocation. 2°. The field near the ends of the pile-up in the case of a source near the pile-up is the field of a pile-up $N + A_{12}/A$ dislocations. 3°. The dislocation experiences the effect of the image forces on the pile-ups determined by the second term in (3.4) as $x \to x_0$, $y \to y_0$. For example, the image field is

$$\sigma_{kj}^{(i)} \approx \sum_{\mu,\nu=1}^{6} \frac{[1 - \operatorname{sign}(\operatorname{Im} p_\nu)\operatorname{sign}(\operatorname{Im} p_\mu)]\,\omega_\nu Q_{kj}^{(\mu)}}{(p_\nu - p_\mu)\,A y_0} \tag{3.5}$$

for $|p_\nu y_0| \ll \min\{(\Delta^2 - x_0^2)/(2x_0),\ \sqrt{\Delta^2 - x_0^2}\}$

It has been taken into account here that the signs of the root with $\operatorname{Im} p_\nu y > 0$ and < 0 are opposite,hence, all values of p_ν with an imaginary part of opposite sign to p_μ yield a contribution to the image. 4°. Non-zero stress components experience a jump as they pass from one side of the pile-up over to the other.

By using the method elucidated, expressions are easily obtained for other pile-up realizations also. For example, the dislocation density in a pile-up with free ends in a quadratic potential well in the presence of a linear source when

$$\sigma(x) = -\alpha x + \sum_{\nu=1}^{6} \frac{\omega_\nu}{A(x - \zeta_{0\nu})}$$

is given by the expression

$$\rho(x) = \sqrt{(\Delta_2 - x)(x - \Delta_1)}\left\{\frac{\alpha}{\pi} + \sum_{\nu=1}^{6} \frac{i\omega_\nu}{\pi A(x - \zeta_{0\nu})\sqrt{(\Delta_2 - \zeta_{0\nu})(\zeta_{0\nu} - \Delta_1)}}\right\} \tag{3.6}$$

The location of the ends of the pile-up is found from (1.4) and (1.5) as

$$\frac{\alpha(\Delta_2 - \Delta_1)^2}{8} - \sum_{\nu=1}^{6}\left\{\frac{\omega_\nu}{A}\left[1 + i\,\frac{2\zeta_{0\nu} - \Delta_1 - \Delta_2}{2\sqrt{(\Delta_2 - \zeta_{0\nu})(\zeta_{0\nu} - \Delta_1)}}\right]\right\} = N \tag{3.7}$$

$$\frac{(\Delta_2 - \Delta_1)\alpha}{2} - \sum_{\nu=1}^{6} \frac{i\omega_\nu}{A\sqrt{(\Delta_2 - \zeta_{0\nu})(\zeta_{0\nu} - \Delta_1)}} = 0 \tag{3.8}$$

According to (1.9) the stress field of the pile-up is

$$\sigma_{kj} = \sum_{\nu=1}^{6} Q_{kj}^{(\nu)}\alpha\left[\zeta_\nu - i\sqrt{(\Delta_2 - \zeta_\nu)(\zeta_\nu - \Delta_1)}\right] + \sum_{\mu,\nu=1}^{6} \frac{\omega_\mu Q_{kj}^{(\nu)}}{A(\zeta_{0\mu} - \zeta_\nu)} \times \left[1 - \frac{\sqrt{(\Delta_2 - \zeta_\nu)(\zeta_\nu - \Delta_1)}}{\sqrt{(\Delta_2 - \zeta_{0\mu})(\zeta_{0\mu} - \Delta_1)}}\right] \tag{3.9}$$

The special value of the expressions for the stresses near defects with free ends is that they are actually Green's functions for a number of important problems about the kind and stresses of defects of pile-up type. In fact,(3.6) and (3.9) depend linearly on the external field,hence,in the presence of distributed fields which can be represented as the superposition of sources,the effect of a field is the superposition of the effects of the sources. Therefore, for the pile-up in a field

$$\sigma(x) = \sigma^{(r)}(x) + \sum_{\nu=1}^{6} \iint_{\Omega} \frac{\omega_\nu(x_0, y_0)}{x - \zeta_{0\nu}}\,dx_0\,dy_0 \tag{3.10}$$

($\sigma^{(r)}(x)$ is a rational fraction, and Ω is the distribution domain of sources with density ω_ν) we have

$$\rho(x) = \sqrt{(\Delta_2 - x)(x - \Delta_1)}\left\{-\frac{1}{\pi^2}\int_{\Delta_1}^{\Delta_2}\frac{\sigma^{(r)}(\xi)\,d\xi}{(\xi - x)\sqrt{(\Delta_2 - \xi)(\xi - \Delta_1)}} - \right. \quad (3.11)$$

$$\left.\sum_{\nu=1}^{6}\frac{i}{\pi A}\iint_{\Omega}\frac{\omega_\mu(x_0, y_0)\,dx_0dy_0}{(\zeta_{0\nu} - x)\sqrt{(\Delta_2 - \zeta_{0\nu})(\zeta_{0\nu} - \Delta_1)}}\right\}$$

$$\sigma_{kj} = \sigma_{kj}^{(r)} + \sum_{\mu,\nu=1}^{6}\frac{Q_{kj}^{(\nu)}}{A}\iint_{\Omega}\frac{\omega_\mu(x_0, y_0)}{\zeta_{0\mu} - \zeta_\nu}\left[1 - \frac{\sqrt{(\Delta_2 - \zeta_\nu)(\zeta_\nu - \Delta_1)}}{\sqrt{(\Delta_2 - \zeta_{0\mu})(\zeta_{0\mu} - \Delta_1)}}\right]dx_0dy_0 \quad (3.12)$$

Here $\sigma_{kj}^{(r)}$ is determined by means of $\sigma^{(r)}$ according to (1.9), and the location of the ends Δ_1 and Δ_2 is determined, as before, from conditions (1.4) and (1.5).

The expressions written down provide the possibility of investigating the fields around cracks and other defects taking into account the cohesive forces of the edges, the nonlinearity of the medium, and the presence of the relaxation zone. By connecting the cohesive force to the thickness of the defect, the nonlinearity to the magnitude of the stress, and being given definite models of the plastic zone, nonlinear equations describing the system exactly (within the framework of elasticity theory) can be written by using (3.11) and (3.12). However, this is beyond the scope of this paper. Let us note that knowledge of just certain integral characteristics discussed later is needed for the estimates. Let us also note that in some cases $\omega_\nu(x, y)$ can be found from experiment.

The listed effects are often observed near the ends of defects in a sufficiently small domain. For example, let us consider ρ and σ_{kj} near the end Δ_1 under the assumption $\Delta_2 - \Delta_1 \gg |\zeta_\nu - \Delta_1| \gg |\zeta_{0\nu} - \Delta_1|$

$$\rho(x) \simeq \frac{M}{\sqrt{x - \Delta_1}} \quad (3.13)$$

$$\sigma_{kj}(x, y) \simeq -\pi i\sum_{\nu=1}^{6} Q_{kj}^{(\nu)}\frac{M}{\sqrt{\zeta_\nu - \Delta_1}} \quad (3.14)$$

$$M = \sum_{\nu=1}^{6}\frac{i}{\pi A}\iint_{\Omega}\frac{\omega_\nu(x_0, y_0)}{\sqrt{\zeta_{0\nu} - \Delta_1}}\,dx_0dy_0$$

The density and stress singularities near the end are the same as for a clamped defect, however, the coefficients of these singularities equal certain fixed values expressed in terms of M.

Since the crack is described by three systems of dislocations, three coefficients must be introduced for it

$$M^{(\alpha)} = \sum_{\nu=1}^{6}\frac{i}{\pi}(A^{-1})^{(\alpha\beta)}\iint_{\Omega}\frac{\omega_\nu^{(\beta)}(x_0, y_0)}{\sqrt{\zeta_{0\nu} - \Delta_1}}\,dx_0dy_0$$

in three formulas of the type (3.13) for $\rho^{(\alpha)}$, but the stresses are given by the formula

$$\sigma_{kj}(x, y) \simeq -\pi i\sum_{\alpha=1}^{3}\sum_{\nu=1}^{6} Q_{kj}^{(\alpha\nu)}\frac{M^{(\alpha)}}{\sqrt{\zeta_\nu - \Delta_1}}$$

Knowledge just of $\sigma_{mj}n_j$ in this plane is needed to determine the work of crack propagation in its plane, hence, only the $\sigma_{m2}(x, 0)$ with the especially simple form

$$\sigma_{m2}(x, 0) \simeq K_m / \sqrt{\Delta_1 - x}$$

$$K_m = -i \sum_{\alpha=1}^{3} \sum_{\nu=1}^{6} \iint_{\Omega} \frac{b_m^{(\alpha)} \omega_\nu^{(\alpha)} (x_0, y_0)}{\sqrt{\zeta_{0\nu} - \Delta_1}} dx_0 dy_0$$

is ordinarily of interest.

The quantities K_m in which the anisotropy does not enter formally are called the effective stress intensity coefficients. The constants $M^{(\alpha)}$ defined as follows in terms of the K_m

$$M^{(\alpha)} = \sum_{\beta=1}^{3} \frac{1}{\pi} (A^{-1})^{(\alpha\beta)} b_m^{(\beta)} K_m$$

are more convenient for the description of the stress rosettes.

4. The sum of (1.9) or (3.12) over the defects can be used to find the fields of several pile-ups (parallel to Oz as before, but with different development planes and dislocations). Therefore, it is simple to find the stresses if the dislocation density in the defects can be found. Expressions for a system of defects between which the spacing is much greater than their size can be written down in a particularly simple form. In this case, the effect of adjacent pile-ups on that given can be replaced in a first approximation by the effect of concentrated N_k-tuple dislocations, and the dipole and other moments of the defects can be taken into account in the next approximations.

5. In the case of a discrete description of the pile-ups, it also turns out to be possible to write an expression for the stress analytically. Ideed, if construction of the polynomial

$$f(x, t) = \prod_{\pi=1}^{N} (x - x_\pi(t)) \tag{5.1}$$

whose roots $x_\pi(t)$ yield the location of the dislocations, is possible, we will have at the time t

$$\frac{f'_{,x}}{f} = \sum_{\pi=1}^{N} \frac{1}{x - x_\pi}$$

(the prime denotes the derivative with respect to x). Hence for a pile-up

$$\sigma_{kj}(x, y) = \sum_{\nu=1}^{6} Q_{kj}^{(\nu)} \frac{f'(x + p_\nu y)}{f(x + p_\nu y)} \tag{5.2}$$

The appropriate polynomials have been found for a number of characteristic stress fields by using the Eshelby, Frank and Nabarro method in both the cases of equilibrium [16, 17] pile-ups and those moving at subsonic velocities [18 – 20]. In both cases (5.2) yields an expression for the stress.

Appendix. Some formulas from six-dimensional theory. Let us present the notation and some results from six-dimensional theory [1, 13, 14] for dislocations-linear forces (linear sources).

Let us introduce the following notation for a 3×3 matrix:

$$(a, b)_{jk} = a_l c_{ljkm} b_m \tag{A.1}$$

Here $c_{ljkm} = c_{jlkm} = c_{kmlj}$ is the elastic modulus matrix, $\mathbf{a}$, $\mathbf{b}$ are vectors selected from the right-hand triple of directions $\{\mathbf{m}, \mathbf{n}, \boldsymbol{\tau}\}$ connected with the source as follows (compare Sect. 3): $\mathbf{n}$ is one of the normals to the source (normal to the pile-up in this case), $\boldsymbol{\tau}$ is parallel to the source, and $\mathbf{m}$ is the second normal to the source.

The stress and displacement fields around linear sources are defined in terms of the eigenvectors and eigenvalues of the following eigenvalue problem in six-dimensional space (over the complex number field):

$$(N - p_\nu I)\xi^{(\nu)} = 0 \tag{A.2}$$

where I is the unit operator and N is formed as follows from the matrix (A.1):

$$N = -\begin{Vmatrix} (n, n)^{-1}(n, m) & (n, n)^{-1} \\ (m, n)(n, n)^{-1}(n, m) & (m, n)(n, n)^{-1} \end{Vmatrix} \tag{A.3}$$

Hence, the eigenvalues p_ν are solutions of an algebraic equation of sixth degree

$$\operatorname{Det}(N - pI) = 0 \tag{A.4}$$

and in the absence of degeneration the eigenvectors $\xi^{(\nu)}$ are proportional to

$$\xi_j^{(\nu)} = \left. \frac{C^{(\nu)} \{Adj\,(N - pI)\}_{j1}}{(\partial/\partial p)\operatorname{Det}(N - pI)} \right|_{p=p_\nu} \tag{A.5}$$

Here $\{AdjN\}_{jk}$ are cofactors of the element N_{jk} of the matrix N and $C^{(\nu)}$ are arbitrary multipliers.

If the first three terms of $\xi^{(\nu)}$ correspond to the complex three-dimensional vector $\mathbf{A}^{(\nu)}$ and the rest to the vector $\mathbf{L}^{(\nu)}$ such that

$$\xi^{(\nu)} = \begin{Vmatrix} \mathbf{A}^{(\nu)} \\ \mathbf{L}^{(\nu)} \end{Vmatrix} \tag{A.6}$$

and the constants $C^{(\nu)}$ are determined from the normalization condition

$$\mathbf{A}^{(\nu)}\mathbf{L}^{(\mu)} + \mathbf{A}^{(\mu)}\mathbf{L}^{(\nu)} = \delta_{\mu\nu} \tag{A.7}$$

then we have

$$\sigma_{kj}(\mathbf{r}_0, \mathbf{r}) = \sum_{\nu=1}^{6} \frac{\operatorname{sign}(\operatorname{Im} p_\nu)\, c_{kjst} A_s^{(\nu)} (L_l^{(\nu)} b_l + A_l^{(\nu)} f_l)(m_t + p_\nu n_t)}{2\pi i \{\mathbf{m}(\mathbf{r} - \mathbf{r}_0) + p_\nu \mathbf{n}(\mathbf{r} - \mathbf{r}_0)\}} \tag{A.8}$$

making (1.1) and (3.1) specific, for the stresses at a point $\mathbf{r}$ due to the source (which is a dislocation with Burgers vector $\mathbf{b}$ and simultaneously a linear force of intensity $\mathbf{f}$) passing through the point $\mathbf{r}_0$.

Recalling (2.3) for $A^{(\alpha\beta)}$, we have

$$A^{(\alpha\beta)} = -\frac{1}{2\pi i} \sum_{\nu=1}^{6} \operatorname{sign}(\operatorname{Im} p_\nu)\, L_s^{(\nu)} L_t^{(\nu)} b_s^{(\alpha)} b_t^{(\beta)} = 2B_{st} b_s^{(\alpha)} b_t^{(\beta)} \tag{A.9}$$

where B_{st} is a non-singular matrix of the coefficients before the logarithms in the expressions for the energy of a unit dislocation. Hence, because the $b^{(\alpha)}$ are not coplanar, the matrix $A^{(\alpha\beta)}$ is also non-singular.

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